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(Research Article)

Three quarter - step block hybrid algorithm for the numerical solution of first order initial value problems of ordinary differential equations

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Abstract

Attracted by the important role of ordinary differential equations in many physical situations like engineering, control theory, biological, and economics, a three quarter - step block hybrid algorithm is constructed for the purpose of solving first-order initial value problems of ordinary differential equations (FIVPODEs). Our approach leverages on off grid points other than the usual whole step point methods constructed by many researchers; this is viewed as an important shift from the usual norm. Some of the advantages of hybrid methods is that they possess remarkably small error constants as observed in Table I among other advantages. Some problems solved by some existing methods are also solved using the constructed algorithm to demonstrate the simplicity, validity and applicability of the method. The results obtained revealed that the method is suitable for all forms of first-order initial value problems be it linear or non-linear ordinary differential equations.

Keywords: Algorithm; Hybrid Block Algorithm; Consistency; Zero stable; Convergence

1. Introduction

The formulation of almost all physical phenomena such as engineering, control theory, biological, and economics are based in differential equations. These equations are important in mathematics and sciences because they can be used to model a wide variety of real-world situations. In physics, for example, differential equations can be used to model the motion of particles in a fluid or the trajectory of a projectile. Differential equations can be used to model the growth of populations or the spread of diseases like bacteria and in the calculation of optimum investment strategies to assist the economist. The ability to model complex situations using differential equations makes them a valuable tool for scientists and engineers. Solving a differential equation enables the scientist to gain a better understanding of how a system behaves and how it might be manipulated to achieve a desired outcome [1, 2, 3, 4, 5, 6]. Additionally, differential equations can be used to predict the future behavior of a system, which can be helpful in designing new technologies or predicting the outcomes of experiments;

A differential equation is a mathematical equation that expresses the rate of change of one variable with respect to another. Differential equations are used to model the motion of objects, the change in population, and the diffusion of molecules [7, 8, 9, 10, 11]. The general form of first-order initial value problems of ordinary differential equations is of the form;

$$\frac{dy(x)}{dx} = f\big(x, y(x)\big), y(x_0) = y_0 \;\; \dots\dots\dots\dots(1)$$

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The most common type of a first-order differential equation is the linear equation which can be solved using the linear equation solving method. Nonlinear first-order differential equations can also be solved, but this is generally more complicated [3, 12, 13].

In this study, we derived a three-quarter-step block hybrid algorithm based on shifted Legendre polynomial basis function with the expectation that the proposed method will give a solution that is as close as possible to the closed form solution for both linear and nonlinear ordinary differential equations. The paper is structured as follows. In Section 2, we derive and analyze the algorithm for consistency, zero stability and convergence. In Section 3, the algorithm is tested on four sample problems of ordinary differential equations. Finally, in section 4, some concluding remarks and suggestions are presented.

2. Derivation of Three Quarter-Step Hybrid Block Method

Equation (1) is approximated using shifted Legendre polynomial of degree $m+q-1$ in the partition $I_n = [a = x_0 < x_1 < x_2 < \cdots < x_n = b]$ of the integration interval $[a, b]$ with a constant step size $h = x_i - x_{i-1}, i = 1, \ldots, n$, by;

$$y(t) = \sum_{i=0}^{m+q-1} c_i \left(\sum_{k=0}^{i} (-1)^{(i+k)} \frac{(i+k)!\, t^k}{(i-k)!\,(k!)^2 1^k} \right) \ldots \ldots \ldots \ldots (2)$$

where $c_i \in \mathbb{R}$, $y \epsilon C^1(a, b)$ and $t = (x - x_n)$. Taking the first derivative of (2) and substituting the result into (1), we obtain a differential system of the form;

$$y'(t) = \sum_{i=0}^{m+q-1} c_i \left(\sum_{k=0}^{i} (-1)^{(i+k)} \frac{(i+k)!\, t^k}{(i-k)!\,(k!)^2 1^k} \right)' \ldots \ldots \ldots \ldots (3)$$

Evaluating (2) at $x_{n+m}, m = 0$ and collocating (3) at $x_{n+q}, q = 0\left(\frac{1}{8}\right)\frac{3}{4}$ where m and q represents interpolation and collocation points respectively, yields the continuous hybrid scheme given below;

$$y(x) = \alpha_0(x) y_n + h \left(\sum_{\tau=0}^{\frac{3}{4}} \beta_{n+\tau}(x) f(x_{n+\tau}, y_{n+\tau}), \tau = \frac{i}{8}, i = 0,1,2, \ldots, 6 \right) \ldots \ldots \ldots (4)$$

Evaluating (4) at the points $x = \frac{1}{8}\left(\frac{1}{4}\right)\frac{3}{4}$, the following discrete algorithms are obtained (see Table I).

Table 1 Discrete algorithms obtained by evaluating (6) for different values of τ

		y_n	hf_n	$hf_{n+\frac{1}{8}}$	$hf_{n+\frac{1}{4}}$	$hf_{n+\frac{3}{8}}$	$hf_{n+\frac{1}{2}}$	$hf_{n+\frac{5}{8}}$	$hf_{n+\frac{3}{4}}$
1. $y_{n+\frac{1}{8}}$	=	1.0	$\frac{19087}{483840}$	$\frac{2713}{20160}$	$-\frac{15487}{161280}$	$\frac{293}{3780}$	$-\frac{6737}{161280}$	$\frac{263}{20160}$	$-\frac{863}{483840}$
2. $y_{n+\frac{1}{4}}$	=	1.0	$\frac{1139}{30240}$	$\frac{47}{252}$	$\frac{11}{10080}$	$\frac{83}{1890}$	$-\frac{269}{10080}$	$\frac{11}{1260}$	$-\frac{37}{30240}$
3. $y_{n+\frac{3}{8}}$	=	1.0	$\frac{137}{3584}$	$\frac{81}{448}$	$\frac{1161}{17920}$	$\frac{17}{140}$	$-\frac{729}{17920}$	$\frac{27}{2240}$	$-\frac{29}{17920}$
4. $y_{n+\frac{1}{2}}$	=	1.0	$\frac{143}{3780}$	$\frac{58}{315}$	$\frac{16}{315}$	$\frac{188}{945}$	$\frac{29}{1260}$	$\frac{2}{315}$	$-\frac{1}{945}$
5. $y_{n+\frac{5}{8}}$	=	1.0	$\frac{3715}{96768}$	$\frac{725}{4032}$	$\frac{2125}{32256}$	$\frac{125}{756}$	$\frac{3875}{32256}$	$\frac{325}{4032}$	$-\frac{275}{96768}$
6. $y_{n+\frac{3}{4}}$	=	1.0	$\frac{41}{1120}$	$\frac{27}{140}$	$\frac{27}{1120}$	$\frac{17}{70}$	$\frac{27}{1120}$	$\frac{27}{140}$	$\frac{41}{1120}$

2.1 Order and Error Constant

Expanding the discrete algorithm in Table 1 in Taylor's series, gives the order and error constants of the discrete schemes in Table 2 below;

Table 2 Order and error constants of each algorithm

Algorithm No	Order	Error Constant
1	$P = 7$	$C_8 = 6.7755 \times 10^{-10}$
2	$P = 7$	$C_8 = 5.0459 \times 10^{-10}$
3	$P = 7$	$C_8 = 5.9871 \times 10^{-10}$
4	$P = 7$	$C_8 = 5.0459 \times 10^{-10}$
5	$P = 7$	$C_8 = 6.7755 \times 10^{-10}$
6	$P = 7$	$C_8 = 0$

2.2 Consistency

According to [14, 15], the discrete algorithm in Table 1 is said to be consistent if the following condition holds:

it has order $\check{p} \geq 1$,

$$\sum_{j=0}^{k} \check{\alpha}_j = 0,$$

$$\sum_{j=0}^{k} j\check{\alpha}_j = \sum_{j=0}^{k} \check{\beta}_j,$$

$$\rho(1) = 0 \text{ and } \rho'(1) = \sigma(1),$$

where $\rho(r)$ and $\sigma(r)$ are the first and the second characteristic polynomials, condition (i) is a sufficient condition for the block discrete algorithms to be consistent since a $\check{p} = 7 \geq 1$**.** Hence, we can assert that the block discrete algorithm is consistent.

2.3 Zero Stability

The block discrete algorithm is said to be zero stable if the roots z_r; $r = 1, \dots, n$ of the first characteristic polynomial $p(z)$, defined by

$$p(z) = det\left| z\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix} \right| = z^5(z - 1) = 0 \Rightarrow z = (0,0,0,0,0,1).$$

Hence, the block method is zero stable, since all roots with modulus one does not have multiplicity exceeding the order of the differential equation in the limit as $h \rightarrow 0$.

2.4 Convergence

From the foregoing, we can safely assert the convergence of the block discrete algorithm since the method is both consistent and zero stable [15, 8].

3. Numerical Experiments

This section considered four examples to illustrate the applicability and effectiveness of the method using Maple 8.1 Software and all graphs were plotted using MATLAB.

3.1 Problem 1

In this problem, we consider the first-order linear initial value problem of ordinary differential equation $y'(x) = x - y$, $y(0) = 0, h = 0.1$ $x \in [0,1]$, the exact solution is $y(x) = x + e^{-x} - 1$. Source: [11] ; the result is shown in Table 3, the two results are compared graphically using solution curve as shown in Figure 1.

Table 3 Comparison of problem 1 results and [11]

x_n	Exact value $y(x_n)$	proposed value y_n	Error in proposed method $e_n = \|y(x_n) - y_n\|$	Error in Sunday et.al (2015) $e_n = \|y(x_n) - y_n\|$
0.1	0.00483741803595957316	0.00483741803595969110	0499×10^{-18}	1.0899×10^{-14}
0.2	0.01873075307798185867	0.018730753077981855325	3.3450×10^{-18}	3.6577×10^{-14}
0.3	0.04081822068171786607	0.040818220681717867201	1.1310×10^{-18}	4.4761×10^{-14}
0.4	0.07032004603563930074	0.070320046035639298765	1.9750×10^{-18}	6.1209×10^{-14}
.5	0.10653065971263342360	0.106530659712633422050	1.5500×10^{-18}	6.1209×10^{-14}
0.6	0.14881163609402643263	0.148811636094026434290	1.6600×10^{-18}	7.0592×10^{-14}
0.7	0.19658530379140951470	0.196585303791409513990	7.1000×10^{-18}	7.9268×10^{-14}
0.8	0.24932896411722159143	0.249328964117221590950	4.8000×10^{-19}	8.3601×10^{-15}
0.9	0.30656965974059911188	0.306569659740599113740	1.8600×10^{-18}	9.4146×10^{-15}
1.0	0.36787944117144232160	0.367879441171442321640	4.0000×10^{-20}	9.7071×10^{-15}

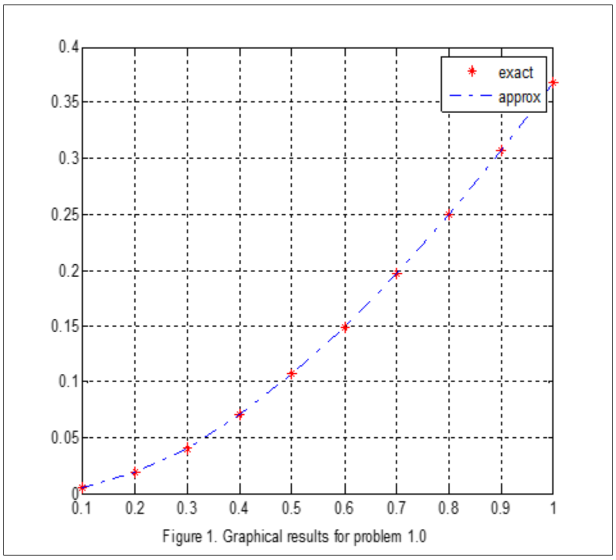

Figure 1 Graphical results for problem 1.0

3.2 Problem 2

Given a first-order nonlinear initial value problem of ordinary differential $y'(x) = xy^3 - y$ with initial condition $y(0) = 1, h = 0.1, x \in [0, 10]$, the exact solution is $y(x) = \frac{2}{\sqrt{2+4x+2e^{2x}}}$. Source: [12]; the result is shown in Table 4. The two results are compared graphically as shown in Figure 2.

Table 4 Problem 2 showing comparison of proposed method and [12]

x_n	**Exact value** $y(x_n)$	**proposed value** y_n	**Error in proposed method** $e_n = \|y(x_n) - y_n\|$	**Error in Turki *et al* (2018)** $e_n = \|y(x_n) - y_n\|$
0.1000	0.90882754533794557710	0.90882754533748953668	4.5604×10^{-13}	8.068236×10^{-7}
0.0500	0.95234404316166747720	0.95234404316067056324	9.9691×10^{-13}	1.852646×10^{-8}
0.0250	0.97560485861385157678	0.97560485861283342993	1.0181×10^{-12}	3.577403×10^{-10}
0.0125	0.98765368982294298440	0.98765368982156289568	1.3801×10^{-12}	6.248941×10^{-12}

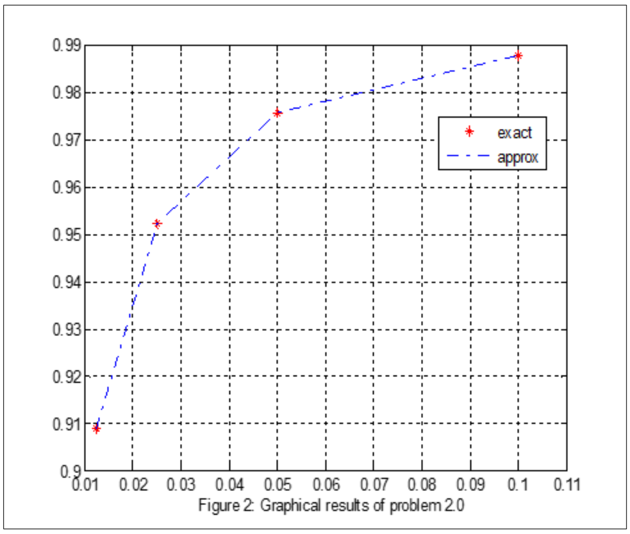

Figure 2 Graphical results for problem 2.0

3.3 Problem 3

Consider a single nonlinear first-order ordinary differential $y'(x) = -10(y-1)^2$ with initial condition $y(0) = 2, h = 0.01, x \in [0, 0.1]$ with exact solution $y(x) = 1 + \frac{1}{1+10x}$. Source: [18]; see Table 5 and Figure 3 for the performance of the algorithm.

Table 5 Comparison results of problem 3 and [18]

x_n	**Exact value** $y(x_n)$	**proposed value** y_n	**Error in proposed method** $e_n = \|y(x_n) - y\|$	**Error in Rufai *et al* (2016)** $e_n = \|y(x_n) - y_n\|$
0.01	1.9090909090909090909	1.9090909090908428697	6.6221×10^{-14}	1.5583×10^{-6}
0.02	1.8333333333333333333	1.8333333333333065608	2.6773×10^{-14}	2.3997×10^{-6}
0.03	1.7692307692307692308	1.7692307692307855098	1.6279×10^{-14}	2.8305×10^{-6}
0.04	1.7142857142857142857	1.7142857142857201935	5.9078×10^{-15}	3.0209×10^{-6}
0.05	1.6666666666666666667	1.6666666666666746849	8.0182×10^{-15}	3.0696×10^{-6}
0.06	1.6250000000000000000	1.6250000000000123149	1.2315×10^{-14}	3.0346×10^{-6}
0.07	1.5882352941176470588	1.5882352941176564918	9.4330×10^{-15}	2.9511×10^{-6}
0.08	1.5555555555555555556	1.5555555555555644109	8.8553×10^{-15}	2.8409×10^{-6}

0.09	1.5263157894736842105	1.5263157894736931814	8.9709×10^{-15}	2.7171×10^{-6}
0.10	1.5000000000000000000	1.5000000000000077453	7.7453×10^{-15}	2.5881×10^{-6}

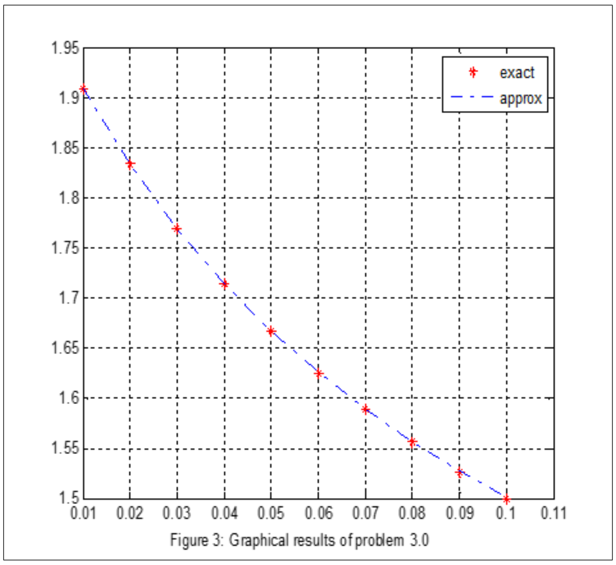

Figure 3 Graphical results for problem 3.0

3.4 Problem 4

We consider the Prothero-Robinson Oscillatory ordinary differential equation $y' = -Ly + cosx - Lsinx, L = -1$, with initial condition $y(0) = 0$ $h = 0.1, x \in [0, 1]$. The exact solution is given by $y(x) = sinx$. Source: [17]; see Table 6 and Figure 4 for the performance of the algorithm

Table 6 Comparison results of problem 4 and [17]

x_n	**Exact value** $y(x_n)$	**proposed value** y_n	**Error in proposed method** $e_n = \lvert y(x_n) - y \rvert$	**Error in Sunday et al (2017)** $e_n = \lvert y(x_n) - y_n \rvert$
0.1	0.099833416646828152307	0.099833416646828151843	4.6400×10^{-19}	5.5511×10^{-17}
0.2	0.19866933079506121546	0.19866933079506121471	7.5000×10^{-19}	1.9429×10^{-16}
0.3	0.29552020666133957511	0.29552020666133957547	3.6000×10^{-19}	4.4409×10^{-16}
0.4	0.38941834230865049167	0.38941834230865049023	1.4400×10^{-18}	9.4369×10^{-16}
0.5	0.47942553860420300027	0.47942553860420299867	1.6000×10^{-18}	1.4433×10^{-15}
0.6	0.56464247339503535720	0.56464247339503535823	1.0300×10^{-18}	1.8874×10^{-15}
0.7	0.64421768723769105367	0.64421768723769105171	1.9600×10^{-18}	2.5535×10^{-15}
0.8	0.71735609089952276163	0.71735609089952275958	2.0500×10^{-18}	3.2196×10^{-15}
0.9	0.78332690962748338846	0.78332690962748339032	1.8600×10^{-18}	3.7748×10^{-15}
1.0	0.84147098480789650665	0.84147098480789650456	2.0900×10^{-18}	4.4409×10^{-15}

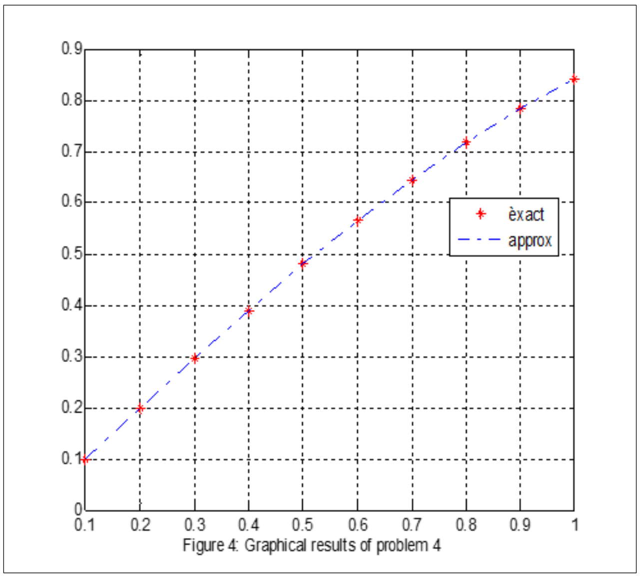

Figure 3 Graphical results for problem 3.0

4. Conclusion

In this study, we derived a three quarter-step block hybrid algorithm for solving first-order linear and nonlinear initial value problems of ordinary differential equations, the method was found to be convergent since it is consistent and zero stable. The method was implemented on some existing sample problems, the numerical results obtained were found to be more accurate when compared with some existing methods contained in the literature herein as shown in Tables 3 - 6 and their respective solution curves (Figures 1- 4). It is important to state that the three quarter-step block hybrid algorithms derived in this research is limited to the solution of first-order IVPs only. Further study could extend this work to the numerical solution of higher order IVPs. The possibility of exploiting other basis function than the shifted Legendre polynomial is also a viable option.

Compliance with ethical standards

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Disclosure of conflict of interest

There is no potential conflict of interest reported by the authors.

Authors' Contributions

This work was carried out in collaboration among the authors. Kamoh, N. M. proposed, derived, and implemented the method. Chun, P. B. analyzed the method while Soomiyol, M. C. presented the results of the method graphically. All authors managed the literature search, read, and approved the final manuscript.

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